SOLUTION OF THE INVERSE CONTACT PROBLEM OF HEAT TRANSFER IN A RECTANGULAR PLATE

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An approximate method of solving an inverse nonlinear contact heat-transfer problem in a rectangular plate by local one-dimensional difference schemes is examined.

Let us consider a two-dimensional contact problem of heat propagation in a rectangular plate $\{0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\}$: Find a continuous function U(x, y, t) satisfying the equations

$$U_t = \operatorname{div}[\lambda_1(U)\operatorname{grad} U] + f_1(U), \quad (x,\ y) \in D_1, \tag{1}$$

$$U_t = \operatorname{div}[\lambda_2(U)\operatorname{grad} U] + f_2(U), \quad (x,\ y) \in D_2, \tag{2}$$

the initial conditions

$$U(x,\ y,\ 0) = \varphi(x,\ y), \tag{3}$$

the boundary conditions of the first kind on $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$ (see Fig. 1)

$$U|_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5} = \psi \tag{4}$$

and matching conditions on the boundary $\Gamma: y = \xi(x)$ of the contact

$$[U]_{y=\xi(x)} = 0, \quad \left[\lambda(U)\frac{\partial U}{\partial n}\right]_{y=\xi(x)} = 0. \tag{5}$$

We assume that the temperature field on the contact line

$$U|_{y=\xi(x)} = U^*(x,\ \xi(x),\ t) \tag{6}$$

or on the line x = c

$$U|_{x=c} = U^*(c,\ y,\ t), \quad 0 < c < a$$

is known from the experiment. It is required to determine the value of the temperature on the boundary Γ_6 and the thermal field at inner points of the plate.

We solve the problem (1)-(6) in two stages. We assume that condition (6) is satisfied on the boundary $y = \xi(x)$. We first solve the direct single-phase problem (1), (3) with boundary conditions of the first kind

$$U|_{\Gamma_1+\Gamma_2+\Gamma_3} = \psi, \quad U|_{y=\xi} = U^*(x,\ \xi(x),\ t)$$

in the domain $D_2 + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma$ and we find $U(x,\ y,\ t)$ and $\lambda_2(U)\left.\frac{\partial U}{\partial n}\right|_{y=\xi+0}$ in D_2. We solve the inverse single-phase problem in the domain $D_1 + \Gamma_4 + \Gamma_5 + \Gamma_6 + \Gamma$: Find the thermal field in D_1 and the value of U(0, y, t) from the conditions (2), (3), $U|_{\Gamma_4+\Gamma_5} = \psi$, $U|_{\xi} = U^*(x, \xi(x), t)$ and

$$\lambda_1(U)\left.\frac{\partial U}{\partial n}\right|_{y=\xi-0} = \lambda_2(U)\left.\frac{\partial U}{\partial n}\right|_{y=\xi+0}. \tag{7}$$

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Locally one-dimensional difference schemes of the fractional step method (schemes of variable directions) are used to solve the formulated problems [1]. They are realized well not only on digital computers but also on R-networks [2].

Let us consider the network $x = x_i$, $y = y_j$ ($i = 0, 1, \ldots, N$; $j = 0, \ldots, M$). Let the lines $x = x_i$, $y = y_j$ intersect the boundary $y = \xi(x)$ only at network nodes and let $N\Delta x = b$. Without limiting the generality, it can be considered that the spacing Δx is constant, $\xi(d) = i_0\Delta x$. Let Δt denote the time spacing and let $U(x_i, y_j, t_n) \equiv U^n_{ij}$. We shall seek the solution of the problem (1)-(6) in the interval $0 \leqslant t \leqslant T$.

Let us introduce the different operators

$$\Delta_x U^n_{ij} = \frac{1}{\Delta x^2}\left[\lambda^{n-1}_{ij}\left(U^n_{i+1,j} - U^n_{ij}\right) - \lambda^{n-1}_{i-1,j}\left(U^n_{ij} - U^n_{i-1,j}\right)\right],$$

$$\Delta_y U^n_{ij} = \frac{2}{\Delta y_{j-1} + \Delta y_j}\left[\lambda^{n-1}_{ij}\,\frac{U^n_{i,j+1} - U^n_{ij}}{\Delta y_j} - \lambda^{n-1}_{i,j-1}\,\frac{U^n_{ij} - U^n_{i,j-1}}{\Delta y_{j-1}}\right].$$

We construct a variable directions scheme. In the domain D_2 we consider the problem

$$\frac{V_{ij} - U^n_{ij}}{\Delta t} = \Delta_x V_{ij} + \frac{f^n_{ij}}{2};\ j = 0, \ldots, M;\ i = i_0 + j + 1, \ldots, N,$$
$$V_{i_0+j,j} = U^{*\left(n+\frac{1}{2}\right)}_{i_0+j,j},\quad V_{Nj} = U^{n+\frac{1}{2}}_{Nj} = \psi_{Nj}. \tag{8}$$

From condition (8) we determine the values of V_{ij} on all the horizontal layers including $j = 0$ and $j = M$ in the domain D_2. Taken as boundary values are

$$U^{*\left(n+\frac{1}{2}\right)}_{i_0+j,j} = U^*\left(x_{i_0+j},\ y_j,\ t_n + \frac{\Delta t}{2}\right),\quad V_{Nj} = U\left(x_N,\ y_j,\ t_n + \frac{\Delta t}{2}\right),$$

and we take the temperature values found in the preceding time layer as U^n_{ij}.

Using the values found for V_{ij} in D_2, we determine V_{ij} in D_1 from the conditions

$$\frac{V_{ji} - U_{ij}}{\Delta t} = \Delta_x V_{ij} + \frac{f^n_{ij}}{2};\ i = 1, \ldots, i_0 + j - 1;\ j = 0, \ldots, M,$$
$$V_{i_0+j,j} = U^{*\left(n+\frac{1}{2}\right)}_{i_0+j,j},$$
$$(\lambda_1)^n_{i_0+j-1,j}\left(U^*_{i_0+j,j} - V_{i_0+j-1,j}\right) = (\lambda_2)^n_{i_0+j+1,j}\left(V_{i_0+j+1,j} - U^*_{i_0+j,j}\right). \tag{9}$$

From (9) we find V_{ij} in D_1 at all the network nodes including the nodes with $i = 0$, i.e., V_{0j}, whereupon we have the values of the auxiliary function V_{ij} in all the inner and boundary nodes of the network. They are obtained by using one-dimensional schemes constructed with the variation of U in the direction of the x axis taken into account. We now solve the direct heat-conduction problem in the domain $D_1 + D_2 + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$,

$$\frac{U^{n+1}_{ij} - V_{ij}}{\Delta t} = \Delta_y U^{n+1}_{ij} + \frac{f_{ij}}{2};\ i = 1, \ldots, N-1;\ j = 1, \ldots, M-1,$$
$$U^{n+1}_{ij}\big|_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5} = \psi,\quad U^{n+1}_{i_0+j,j} = U^*_{i_0+j,j}. \tag{10}$$

From (10) we determine U^{n+1}_{ij} at all network nodes in the vertical layers, including the value U^{n+1}_{0j} which is the approximate solution of the initial inverse problem (1)-(6) for $t = (n + 1)\cdot\Delta t$.

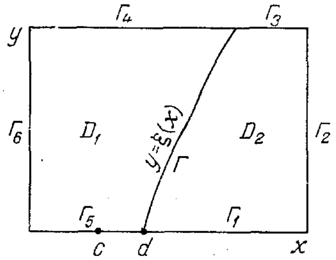

Fig. 1. Structure of the inverse heat-transfer problem domain.

The difference scheme used is absolutely stable for the solution of direct problems [3]. For the solution of the inverse problem (9) in the domain D_1 the system matrix turns out to be poorly specified, a cumulative error occurs, the result is distorted, i.e., the problem turns out to be incorrect. Smoothing of the input information, as well as smoothing of the results on each time layer, if necessary, are performed to suppress the instability.

In our case the initial errors can occur because of inaccurate experimental determination of U^*, as well as in the determination of $V_{i_0+j-1,j}$ at the near-boundary (from the left) nodes from the condition of equality of the fluxes in (9). Let us show how this smoothing can be performed for $V_{i_0+j-1,j}$, say. We construct the functional

$$\Phi(\alpha)=\sum_{j=0}^{M}(T_j-V_{i_0+j-1,j})^2+\frac{\alpha}{\Delta y^2}\sum_{j=1}^{M}(T_j-T_{j-1})^2+\frac{\alpha}{\Delta y^4}\sum_{j=1}^{M}(T_{j+1}-2T_j+T_{j-1})^2.$$

We seek the minimum $\Phi(\alpha)$ in $T_0, \ldots, T_M$. Differentiating with respect to T_j, we obtain a linear system of the form $AT = F$ to determine $T = \{T_0, \ldots, T_M\}$ as a function of α. To determine the optimal value of the regularization parameter α, we construct the following iteration process.

Let δ be the error in approximating the difference scheme ($\delta = o(\Delta x + \Delta y + \Delta t)$). We consider the equation

$$\Big[\sum_{j=0}^{M}(T_j-V_{i_0+j-1,j})^2\Big]^{1/2}-\delta=0. \tag{11}$$

We determine α from (11). For convenience of the solution, we follow [4] and write (11) in the form

$$g(p)\equiv\Big[\sum_{j=0}^{M}(T_j-V_{i_0+j-1,j})^2\Big]^{-\frac{1}{2}}-\frac{1}{\delta}=0,$$

where $p = 1/\alpha$.

The iteration process to determine α is constructed by the Newton method

$$p_{k+1}=p_k-\frac{g(p_k)}{g'(p_k)}, \tag{12}$$

$$g'(p_k)=-\frac{1}{p^2}\,\frac{g\left(\dfrac{1}{\alpha+\Delta\alpha}\right)-g\left(\dfrac{1}{\alpha-\Delta\alpha}\right)}{2\Delta\alpha}. \tag{13}$$

The algorithm of the regularization process is the following:

1) we select the zeroth approximation $p = p_0$;

2) we solve the system $AT = F$ at $p = p_0$. We find T_j^0 and $g(1/\alpha_0)$. We solve the system $AT = F$ at $p = 1/(\alpha_0 + \Delta\alpha)$ and find $g\left(\dfrac{1}{\alpha_0+\Delta\alpha}\right)$. We find $\left(\dfrac{1}{\alpha_0-\Delta\alpha}\right)$ analogously;

TABLE 1. Values of the Boundary Function U(x, 0, t)

$y=0,0$	$y=0,2$	$y=0,4$	$y=0,6$	$y=0,8$	$y=1,0$
5,4697	5,4397	5,3986	5,3613	5,3264	5,2900
5,4310	5,4100	5,3830	5,3740	5,3440	5,3160
5,4691	5,4367	5,3927	5,3600	5,3172	5,2873
5,4772	5,4406	5,4037	5,3606	5,3282	5,2915
$t=10$					
7,0699	7,0404	6,9814	6,9715	6,9516	6,9004
7,0450	7,0200	6,9870	6,9710	6,9360	6,9070
7,0689	7,0392	6,9823	6,9584	6,9477	6,8964
7,0711	7,0427	7,0143	6,9857	6,9570	6,9282

3) we find $g'(p_0)$ from (13);

4) we calculate p_1 from (12);

5) substituting p_1 into $AT = F$, we find the first approximation T_j^1 and repeat 1)-5).

Having determined T_j, we set $V_{i_0+j-1,j} = T_j$ and continue to solve the inverse problem in the domain D_1.

We illustrate the method elucidated by the problem (1)-(6) in which

$$\lambda_1(U) = U^2, \quad \lambda_2(U) = \frac{4}{9}U^2, \quad f_1(U) \equiv 0, \quad f_2(U) = \frac{3}{4U}, \quad \xi(x) = 2x,$$

$$\varphi(x, y) = \begin{cases} (10 - 2x - 2y)^{\frac{1}{2}}, & y > 2x, \\ (10 - 3x - 1.5\,y)^{\frac{1}{2}}, & y < 2x, \end{cases}$$

$$U_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5} = \begin{cases} (10 - 3x + 4t)^{\frac{1}{2}}; \; y = 0; \; 0 \leqslant x \leqslant 0.5, \\ (8.5 - 15y + 4t)^{\frac{1}{2}}; \; x = 0.5; \; 0 \leqslant y \leqslant 1, \\ (8 - 2x + 4t)^{\frac{1}{2}}; \; y = 1; \; 0 \leqslant x \leqslant 0.5 \end{cases}$$

with the additional condition

$$U|_{y=2x} = U^* = (10 - 6x + 4t)^{\frac{1}{2}}.$$

Presented in Table 1 are results of computations on an M-222 digital computer, on an R-network, and values of the exact solution (first, second, and fourth rows, respectively) for $\Delta x = 0.1$; $\Delta y = 0.2$; $\Delta t = 0.1$. No instability in the inverse problem is observed for these spacings. Presented in the third row are values of the regularized solution for $\Delta x = 0.05$; $\Delta y = 0.1$; $\Delta t = 0.01$ for which an instability in the solution of the inverse problem has already been observed explicitly.

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